

$U(1)$ Invariant $F(\tilde{R})$ Hořava-Lifshitz Gravity

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ABSTRACT: This paper is devoted to the study of various aspects of projectable $F(R)$ Hořava-Lifshitz (HL) gravity. We show that some versions of $F(R)$ HL gravity may have stable de Sitter solution and unstable flat space solution. In this case, the problem of scalar graviton does not appear because flat space is not vacuum state. Generalizing the $U(1)$ HL theory proposed in arXiv:1007.2410, we formulate $U(1)$ extension of scalar theory and of $F(R)$ Hořava-Lifshitz gravity. The Hamiltonian approach for such the theory is developed in full detail. It is demonstrated that its Hamiltonian structure is the same as for the non-relativistic covariant HL gravity. The spectrum analysis performed around flat background indicates towards the consistency of the theory because it contains graviton with only transverse polarization. Finally, we analyze the spatially-flat FRW equations for $U(1)$ invariant $F(R)$ Hořava-Lifshitz gravity.

KEYWORDS: Hořava-Lifshitz gravity, Hamiltonian structure, $F(R)$ gravity, stability.

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1. Introduction

In 2009 Petr Hořava formulated new proposal of quantum theory of gravity that is power counting renormalizable [1]. This theory is now known as Hořava-Lifshitz gravity (HL gravity). It was also expected that this theory reduces to General Relativity in the infrared (IR) limit. The HL gravity is a new and intriguing formulation of gravity as a theory with reduced amount of symmetries and this fact leads to remarkable new phenomena ¹.

The HL gravity is based on an idea that the Lorentz symmetry is restored in IR limit of given theory and can be absent at high energy regime of given theory. Explicitly, Hořava considered systems whose scaling at short distances exhibits a strong anisotropy between space and time,

$$\boldsymbol{x}' = l\boldsymbol{x}, \quad t' = l^z t. \quad (1.1)$$

In $(D+1)$ dimensional space-time in order to have power counting renormalizable theory requires that $z \geq D$. It turns out however that the symmetry group of given theory

¹For review and extensive list of references, see [2].

is reduced from the full diffeomorphism invariance of General Relativity to the foliation preserving diffeomorphism

$$x'^i = x^i + \zeta^i(t, \mathbf{x}) , \quad t' = t + f(t) . \quad (1.2)$$

The common property of all modified theories of gravity is that whenever the group of symmetries is restricted (as for example 1.2) one more degree of freedom appears that is a spin-0 graviton. An existence of this mode could be dangerous for all these theories (for review, see [4]). For example, in order to have the theory compatible with observations one has to demand that this scalar mode decouples in the IR regime. Unfortunately, it seems that this might not be the case. It was shown that the spin-0 mode is not stable in the original version of the HL theory [1] as well as in SVW generalization [5]. Note that in both of these two versions, it was all assumed the projectability condition that means that the lapse function N depends on t only. This assumption has a fundamental consequence for the formulation of the theory since there is no local form of the Hamiltonian constraint but the only global one. Even if these instabilities indicate to problems with the projectable version of the HL theory it turns out that this is not the end of the whole story. Explicitly, these instabilities are all found around the Minkowski background. Recently, it was indicated that the de Sitter space-time is stable in the SVW setup [6] and hence it seems to be reasonable to consider de Sitter background as the natural vacuum of projectable version of the HL gravity. This may be especially important for the theories with unstable flat space solution.

On the other hand there is the second version of the HL gravity where the projectability condition is not imposed so that $N = N(\mathbf{x}, t)$. Properties of such theory were extensively studied in [7]. It was shown recently in [8] that so called healthy extended version of such theory could really be an interesting candidate for the quantum theory of reality without ghosts and without strong coupling problem despite its unusual Hamiltonian structure [9]. Nevertheless, such theory is not free from its own internal problems.

Recently Hořava and Melby-Thompson [10] proposed very interesting way to eliminate the spin-0 graviton. They considered the projectable version of the HL gravity together with extension of the foliation preserving diffeomorphism to include a local $U(1)$ symmetry. The resulting theory is then called as non-relativistic covariant theory of gravity ². It was argued there [10] that the presence of this new symmetry forces the coupling constant λ to be equal to one. However, this result was questioned in [13] (see also [12, 14]) where an alternative formulation of non-relativistic general covariant theory of gravity was presented. Furthermore, it was shown in [10, 13] that the presence of this new symmetry implies that the spin-0 graviton becomes non-propagating and the spectrum of the linear fluctuations around the background solution coincides with the fluctuation spectrum of General Relativity. This construction was also extended to the case of RFDiff invariant HL gravities [8, 15] in [14] where it was shown that the number of physical degrees of freedom coincides with the number of physical degrees of freedom in General Relativity.

²This theory was also studied in [11, 12, 13, 14].

The goal of this paper is to extend above construction to the case of $F(\tilde{R})$ HL gravities. These models were developed in series of papers [16]³. $F(\tilde{R})$ HL gravity can be considered as natural generalization of covariant $F(R)$ gravity. Current interest to $F(R)$ gravity is caused by several important reasons. First of all, it is known that such theory may give the unified description of the early-time inflation and late-time acceleration (for a review, see [17, 21].) Moreover, the whole sequence of the universe evolution epochs: Inflation, radiation/matter dominance and dark energy may be obtained within such theory. The remaining freedom in the choice of $F(R)$ function could be used for fitting the theory with observational data. Second, it is known that higher derivatives gravity (like R^2 -gravity, for a review, see [22]) has better ultraviolet behavior than conventional General Relativity. Third, modified gravity is pretending also to be the gravitational alternative for Dark Matter. Fourth, it is expected that consistent quantum gravity emerging from string/M-theory should be different from General Relativity. Hence, it should be modified by fundamental theory. Of course, all these reasons remain to be the same also for the HL gravity. Additionally, it is expected that such modification may be helpful for resolution of internal inconsistency problems of the HL theory. Indeed, we will present the example of $F(\tilde{R})$ HL gravity which has stable de Sitter solution but unstable flat space solution. In such a case, the original scalar graviton problem formally disappears because flat space is not vacuum state. Hence, there is no sense to study propagators structure around flat space. The complete propagators structure should be investigated around de Sitter solution which seems to be the candidate for vacuum space.

The paper is organized as follows. In the next section, we briefly review the construction of $F(\tilde{R})$ HL gravity. Section 3 is devoted to study the de Sitter solutions and flat space solutions in such theory. Their stability is analyzed and it is shown that some versions of the theory may have stable de Sitter but unstable flat space solution. Clearly, this is indication that for such theories the appearance of scalar graviton is not a problem due to fact that flat space is not vacuum solution. The whole spectrum analysis should be developed around de Sitter vacuum. $U(1)$ extension of $F(\tilde{R})$ HL gravity as well as of scalar HL theory is given in section 4. To check the consistency of such construction, two alternative approaches to such extension are proposed. The Hamiltonian structure of $U(1)$ invariant $F(\tilde{R})$ HL gravity is carefully investigated in section 5. It is demonstrated that its Hamiltonian structure is the same as in non-relativistic covariant HL gravity. We also argue on the general grounds of the Hamiltonian formalism of constrained system that the number of physical degrees of freedom coincides with the number of physical degrees of freedom in $F(R)$ gravity despite of the absence of the local Hamiltonian constraint. It is shown that for the special case $F(x) = x$ the Hamiltonian structure of $U(1)$ invariant $F(\tilde{R})$ gravity coincides with the Hamiltonian structure of non-relativistic covariant HL gravity found in [14] which can be considered as a nice check of our analysis. The fluctuations around flat background for $U(1)$ invariant $F(R)$ are studied in section 6. It is explicitly demonstrated that perturbations spectrum contains the graviton with only transverse polarization. This indicates that scalar graviton problem may be solved within such theory.

³For further study in given direction, see [9, 18], and for review, see [17].

Section 7 is devoted to demonstration that spatially-flat FRW solutions of $U(1)$ invariant $F(\tilde{R})$ gravity coincide with the ones for the same theory without $U(1)$ invariance. Some summary and outlook are given in last section.

2. Brief Review of $F(\tilde{R})$ HL Gravity

In this section we give a brief review of $F(\tilde{R})$ HL gravity (for more extensive review, see [17]). This theory is naturally formulated in ADM formulation of gravity [3]. Let us consider $D + 1$ dimensional manifold \mathcal{M} with the coordinates x^μ , ($\mu = 0, \dots, D$) and where $x^\mu = (t, \mathbf{x})$, $\mathbf{x} = (x^1, \dots, x^D)$. We assume that this space-time is endowed with the metric $\hat{g}_{\mu\nu}(x^\rho)$ with signature $(-, +, \dots, +)$. Suppose that \mathcal{M} can be foliated by a family of space-like surfaces Σ_t defined by $t = x^0$. Let g_{ij} , ($i, j = 1, \dots, D$) denotes the metric on Σ_t with inverse g^{ij} so that $g_{ij}g^{jk} = \delta_i^k$. We further introduce the operator ∇_i that is covariant derivative defined with the metric g_{ij} . We introduce the future-pointing unit normal vector n^μ to the surface Σ_t . In ADM variables one has $n^0 = \sqrt{-\hat{g}^{00}}$, $n^i = -\hat{g}^{0i}/\sqrt{-\hat{g}^{00}}$. We also define the lapse function $N = 1/\sqrt{-\hat{g}^{00}}$ and the shift function $N^i = -\hat{g}^{0i}/\hat{g}^{00}$. In terms of these variables the components of the metric $\hat{g}_{\mu\nu}$ are written as

$$\begin{aligned}\hat{g}_{00} &= -N^2 + N_i g^{ij} N_j, & \hat{g}_{0i} &= N_i, & \hat{g}_{ij} &= g_{ij}, \\ \hat{g}^{00} &= -\frac{1}{N^2}, & \hat{g}^{0i} &= \frac{N^i}{N^2}, & \hat{g}^{ij} &= g^{ij} - \frac{N^i N^j}{N^2}.\end{aligned}$$

Then it is easy to see that

$$\sqrt{-\det \hat{g}} = N \sqrt{\det g}. \quad (2.1)$$

The extrinsic derivative is defined as

$$K_{ij} = \frac{1}{2N}(\partial_t g_{ij} - \nabla_i N_j - \nabla_j N_i). \quad (2.2)$$

It is well-known that the components of the Riemann tensor can be written in terms of ADM variables. For example, in case of Riemann curvature we have

$$^{(D+1)}R = K^{ij}K_{ij} - K^2 + R + \frac{2}{\sqrt{-\hat{g}}}\partial_\mu \left(\sqrt{-\hat{g}} n^\mu K \right) - \frac{2}{\sqrt{\hat{g}}N}\partial_i \left(\sqrt{\hat{g}} g^{ij} \partial_j N \right), \quad (2.3)$$

where R is D -dimensional curvature. The general formulation of Hořava-Lifshitz $F(\tilde{R})$ gravity was presented in series of papers in [16]⁴. This construction is based on the modification of the relation (2.3). In fact, the action introduced in [16] takes the form

$$S_{F(\tilde{R})} = \frac{1}{\kappa^2} \int dt d^D \mathbf{x} \sqrt{\hat{g}} N F(\tilde{R}), \quad (2.4)$$

where

$$\tilde{R} = K_{ij} \mathcal{G}^{ijkl} K_{kl} + \frac{2\mu}{\sqrt{-\hat{g}}}\partial_\mu \left(\sqrt{-\hat{g}} n^\mu K \right) - \frac{2\mu}{\sqrt{\hat{g}}N}\partial_i \left(\sqrt{\hat{g}} g^{ij} \partial_j N \right) - \mathcal{V}(g), \quad (2.5)$$

⁴For further study in given direction, see [9, 18], and for review, see [17]

where μ is constant, $K = K_{ij}g^{ji}$ and where the generalized de Witt metric \mathcal{G}^{ijkl} is defined as

$$\mathcal{G}^{ijkl} = \frac{1}{2}(g^{ik}g^{jl} + g^{il}g^{jk}) - \lambda g^{ij}g^{kl}, \quad (2.6)$$

where λ is real constant. Finally $\mathcal{V}(g)$ is general function of g_{ij} and its covariant derivatives. We should also note that the special case of $F(\tilde{R})$ HL gravity where $\mu = 0$ was introduced in [19]. An important drawback of the case $\mu = 0$ is that it cannot lead to the FRW cosmological equations directly. The FRW equations may be obtained only as the limit $\mu = 0$ from general theory. In fact, due to the absence of the derivative terms in (2.5) this theory cannot flow to $F(R)$ gravity action in IR.

We conclude this section with the remark that the action (2.4) is invariant under restricted group of symmetries with is foliation preserving diffeomorphism

$$t' - t = f(t), \quad x'^i - x^i = \xi^i(t, \mathbf{x}). \quad (2.7)$$

Note that under this transformations the metric components transform as

$$\begin{aligned} N'_i(\mathbf{x}', t') &= N_i(\mathbf{x}, t) - N_i(\mathbf{x}, t)\dot{f} - N_j(\mathbf{x}, t)\partial_i\xi^j(\mathbf{x}, t) - g_{ij}(\mathbf{x}, t)\dot{\xi}^j(\mathbf{x}, t), \\ N'(t') &= N(t) - N(t)\dot{f}, \\ g'_{ij}(\mathbf{x}', t') &= g_{ij}(\mathbf{x}, t) - g_{ik}(\mathbf{x}, t)\partial_j\xi^k(\mathbf{x}, t) - \partial_i\xi^k(\mathbf{x}, t)g_{kj}(\mathbf{x}, t). \end{aligned}$$

3. Solutions and their stability in $F(\tilde{R})$ HL gravity

In this section, we consider different solutions of standard $F(\tilde{R})$ Hořava-Lifshitz gravity, specially de Sitter and vacuum solutions. The stability of this kind of solutions is studied, and it is shown that it depends completely on the choice of function $F(\tilde{R})$. This suggests the way to resolve the scalar graviton problem of Hořava-Lifshitz gravity: Its $F(\tilde{R})$ version may have stable de Sitter vacuum but not flat-space which turns out to be unstable. In this situation, the problem is solved simply due to the fact that space flat is not vacuum. In order to study the consistency of the theory its spectrum in de Sitter space should be investigated what lies beyond the scopes of this work.

3.1 Stability of de Sitter solutions in $F(\tilde{R})$ gravity

Let us consider the stability of the de Sitter solution. As dark energy and even inflation may be described (in their simplest form) by the de Sitter space, its stability becomes very important topic. Especially in the case of inflation, where a graceful exit is needed to enter to radiation/matter dominance, de Sitter space should be unstable. In general, standard $F(R)$ gravity contains several de Sitter points, which represent critical points (see [23]). This analysis can be extended to $F(\tilde{R})$ Hořava-Lifshitz gravity. Let us write the first FRW equation in $F(\tilde{R})$ Hořava-Lifshitz gravity [16]

$$0 = F(\tilde{R}) - 6 \left[(1 - 3\lambda + 3\mu)H^2 + \mu\dot{H} \right] F'(\tilde{R}) + 6\mu H \dot{R} F''(\tilde{R}) - \kappa^2 \rho_m - \frac{C}{a^3}, \quad (3.1)$$

For a given $F(\tilde{R})$, de Sitter solution $H(t) = H_0$, where H_0 being a constant, has to satisfy the first equation FRW equation (3.1),

$$0 = F(\tilde{R}_0) - 6H_0^2(1 - 3\lambda + 3\mu)F'(\tilde{R}_0), \quad (3.2)$$

where $C = 0$ and it is assumed the absence of any kind of matter. The scalar \tilde{R} is given in this case by,

$$\tilde{R}_0 = 3(1 - 3\lambda + 6\mu)H_0^2. \quad (3.3)$$

Then, the positive roots of equation (3.2) are the de Sitter points allowed by a particular choice of an $F(\tilde{R})$ function. Assuming de Sitter solution, one can write $F(\tilde{R})$ around \tilde{R}_0 as a series,

$$F(\tilde{R}) = F_0 + F'_0(\tilde{R} - \tilde{R}_0) + \frac{F''_0}{2}(\tilde{R} - \tilde{R}_0)^2 + \frac{F_0^{(3)}}{6}(\tilde{R} - \tilde{R}_0)^3 + O(\tilde{R}^4). \quad (3.4)$$

Here, the primes denote derivative with respect to \tilde{R} while the subscript 0 means that it is evaluated in \tilde{R}_0 . Then, we can perturb the solution writing the Hubble parameter as,

$$H(t) = H_0 + \delta(t). \quad (3.5)$$

Using the function $F(\tilde{R})$ evaluated around a given de Sitter solution (3.4), and the perturbed solution (3.5) in the first FRW equation (3.1), it yields,

$$\begin{aligned} 0 = & \frac{1}{2}F_0 - 3H_0^2(1 - 3\lambda + 3\mu) \\ & - 3H_0 \left[((1 - 3\lambda)F'_0 + 6F''_0H_0^2(-1 + 3\lambda - 6\mu)(-1 + 3\lambda - 3\mu)) \delta(t) \right. \\ & \left. + 6F''_0\mu H_0(-1 + 3\lambda - 3\mu)\dot{\delta}(t) - 12F''_0\mu^2\ddot{\delta}(t) \right]. \end{aligned} \quad (3.6)$$

Here, we have taken the linear approach on δ and its derivatives. Note that the first two terms in the equation (3.6) can be dropped because of the equation (3.2), which is assumed to be satisfied. Then, equation (3.6) can be written in a more convenient way as,

$$\begin{aligned} & \ddot{\delta}(t) + \frac{H_0(1 - 3\lambda + 9\mu)}{2\mu}\dot{\delta}(t) \\ & + \frac{1}{12\mu^2} \left[(3\lambda - 1)\frac{F'_0}{F''_0} - 6H_0^2(1 - 3\lambda + 6\mu)(1 - 3\lambda + 3\mu) \right] \delta(t) = 0. \end{aligned} \quad (3.7)$$

Then, the perturbations of de Sitter solution will depend completely on the model, that is on the derivatives of the $F(\tilde{R})$ function, as well as on the parameters of the theory (λ, μ) . Note that the instability will be large if the term in front of $\delta(t)$ is negative, as the perturbations will increase exponentially, while if we have a positive frequency, the perturbations will behave as a damped harmonic oscillator. During dark energy epoch, as the scalar curvature is very small, the IR limit of the theory can be assumed, where General Relativity is recovered, and in such a case we have $\lambda = \mu \sim 1$, and the frequency will depend completely on the value of $\frac{F'_0}{F''_0}$. In order to avoid large instabilities during the dark energy phase, the condition $\frac{F'_0}{F''_0} > 12H_0^2$ has to be imposed. Nevertheless, during the

inflationary epoch, where the scalar curvature is large, the IR limit is not a convenient approach, and the perturbations will depend also on the values of (λ, μ) . Although if we assume a very small F_0'' , the first term in the frequency of the equation (3.7) will dominate and if $\lambda > 1/3$, the stability of the solution will depend on the sign of $\frac{F_0'}{F_0''}$, being stable when such a coefficient is positive.

3.2 On flat space solution in $F(\tilde{R})$ gravity

Let us now study flat space solutions in $F(\tilde{R})$ HL gravity. In this section we restrict to the case of $3 + 1$ dimensional space-time. A general metric in the ADM decomposition in a $3 + 1$ space-time is given by,

$$ds^2 = -N^2 dt^2 + g_{ij}^{(3)}(dx^i + N^i dt)(dx^j + N^j dt), \quad (3.8)$$

where $i, j = 1, 2, 3$, N is the so-called lapse variable, and N^i is the shift 3-vector. For flat space the variables from the metric (3.8) take the values,

$$N = 1, \quad N_i = 0 \quad \text{and} \quad g_{ij} = g_0 \delta_{ij}, \quad (3.9)$$

where g_0 is a constant. Then, the scalar curvature $\tilde{R} = 0$, and so that our theory has flat space solution, the function $F(\tilde{R})$ has to satisfy,

$$F(0) = 0. \quad (3.10)$$

Hence, we assume the condition (3.10) is satisfied, otherwise the theory has no flat space solution. We are interested to study the stability of such solutions for a general $F(\tilde{R})$, by perturbing the metric in vacuum (3.9), this yields,

$$N = 1 + \delta_N(t) \quad \text{and} \quad g_{ij} = g_0(1 + \delta_g(t))\delta_{ij}. \quad (3.11)$$

For simplicity, we restrict the study on the time-dependent perturbations (no spatial ones) and on the diagonal terms of the metric. Note that, as we are assuming the projectability condition, by performing a transformation of the time coordinate, we can always rewrite $N = 1$. Then, the study of the perturbations is focused on the spatial components of the metric g_{ij} , which can be written in a more convenient way as,

$$g_{ij} = g_0 \left(1 + \int \delta(t) dt \right) \delta_{ij} \sim g_0 e^{\int \delta(t) dt} \delta_{ij}. \quad (3.12)$$

By inserting (3.12) in the first field equation, obtained by the variation of the action on N , it yields at lowest order on $\delta(t)$,

$$12\mu^2 F_0'' \ddot{\delta}(t) \delta(t) - 6\mu^2 F_0'' \dot{\delta}^2(t) + (-1 + 3\lambda) F_0' \delta^2(t) = 0. \quad (3.13)$$

Here the derivatives F_0' , F_0'' are evaluated on $\tilde{R} = 0$. Then, the perturbation δ will depend completely on the kind of theory assumed. We can study some general cases by imposing conditions on the derivatives of $F(\tilde{R})$.

- For the case $F_0'' = 0$, it gives $\delta(t) = 0$, such that at lowest order the flat space is completely stable for this case.
- For $F_0' = 0$ and $F_0'' \neq 0$, the differential equation (3.13) has the solution,

$$\delta(t) = C_1 t \left(1 + \frac{C_1 t}{C_2} \right) + C_2 \quad (3.14)$$

where $C_{1,2}$ are integration constants. Then, for this case, the perturbations grow as the power of the time coordinate, and flat space becomes unstable.

Hence, depending on the theory, the flat space solution will be stable or unstable, which becomes very important as it could be used to distinguish the theories or analyze their consistency.

3.3 A simple example

Let us now discuss a simple example. We consider the function,

$$F(\tilde{R}) = \kappa_0 \tilde{R} + \kappa_1 \tilde{R}^n, \quad (3.15)$$

where $\kappa_{0,1}$ are coupling constants and $n > 1$. Note that this family of theories satisfies the condition (3.10). The values of first and second derivatives evaluated in the solution depend on the value of n in (3.15),

$$F'(0) = \kappa_0, \quad F''(0) = \kappa_1 n(n-1) \tilde{R}^{n-2}. \quad (3.16)$$

Then, we can distinguish between the cases,

- For $n \neq 2$, we have $F''(0) = 0$, and by the analysis performed above, it follows that flat space is stable
- For $n = 2$, we have $F''(0) = \kappa_1$, and flat space is unstable.

Hence, we have shown that the stability of solution depends completely on the details of the theory.

We can now analyze the de Sitter solution. Using the equation (3.2), we can find the de Sitter points allowed by the class of theories given in Eq. (3.15),

$$\frac{3}{2} H_0^2 (3\lambda - 1) \kappa_0 + \frac{\kappa_1 (3H_0^2 (1 - 3\lambda + 6\mu))^n (1 - 3\lambda + 6\mu - 2n(1 - 3\lambda + 3\mu))}{2(1 - 3\lambda + 6\mu)} = 0. \quad (3.17)$$

Resolving the Eq. (3.17), de Sitter solutions are obtained. For simplicity, let us consider $n = 2$, in such a case the equation (3.17) has two roots for H_0 given by,

$$H_0 = \pm \frac{\sqrt{\kappa_0(3\lambda - 1)}}{3\sqrt{\kappa_1(1 - 3\lambda + 6\mu)(-1 + 3\lambda - 2\mu)}}. \quad (3.18)$$

As we are interested in de Sitter points, we just consider the positive root in (3.18). Then, the stability of such de Sitter point can be analyzed by studying the derivatives of the

function $F(\tilde{R})$ evaluated in H_0 . The stability will depend on the value of $\frac{F'_0}{F''_0}$, which for this case yields,

$$\frac{F'_0}{F''_0} = \frac{\kappa_0}{2\kappa_1} + 12H_0^2. \quad (3.19)$$

In the IR limit of the theory ($\lambda \rightarrow 1$, $\mu \rightarrow 1$), the condition for the stability of de Sitter points $\frac{F'_0}{F''_0} > 12H_0^2$ is clearly satisfied by (3.19). Even in the non IR limit, $F''_0 = 2\kappa_1$ and assuming $\kappa_1 \ll 1$, we have that,

$$\frac{F'_0}{F''_0} = 3H_0^2(1 - 3\lambda + 6\mu) + \frac{\kappa_0}{2\kappa_1}. \quad (3.20)$$

As this term is positive, we have that the instabilities will oscillate and be damped, such that the de Sitter point becomes stable.

Thus, we presented the example of the $F(\tilde{R})$ theory where flat space is unstable solution and de Sitter space is stable solution. The problem of scalar graviton does not appear in this theory because one has to analyze the spectrum of theory around de Sitter space which is real vacuum. Indeed, flat space is not stable and cannot be considered as the vacuum solution. Of course, deeper analysis of de Sitter spectrum structure of the theory is necessary. Nevertheless, as we see already standard $F(\tilde{R})$ gravity suggests the way to resolve the pathologies which are well-known in Hořava-Lifshitz gravity.

4. $U(1)$ Invariant $F(\tilde{R})$ Hořava-Lifshitz Gravity

Our goal is to see whether it is possible to extend the gauge symmetries for above action as in [10]. As the first step we introduce two non-dynamical fields A, B and rewrite the action (2.4) into the form

$$S_{F(\tilde{R})} = \frac{1}{\kappa^2} \int dt d^D \mathbf{x} \sqrt{g} N (B(\tilde{R} - A) + F(A)). \quad (4.1)$$

It is easy to see that solving the equation of motion with respect to A, B this action reduces into (2.4). On the other hand when we perform integration by parts we obtain the action in the form

$$\begin{aligned} S_{F(\tilde{R})} = \frac{1}{\kappa^2} \int dt d^D \mathbf{x} \left(\sqrt{g} N B (K_{ij} \mathcal{G}^{ijkl} K_{kl} - \mathcal{V}(g) - A) \right. \\ \left. + \sqrt{g} N F(A) - 2\mu \sqrt{g} N \nabla_n B K + 2\mu \partial_i B \sqrt{g} g^{ij} \partial_j N \right), \end{aligned} \quad (4.2)$$

where we ignored the boundary terms and where

$$\nabla_n B = \frac{1}{N} (\partial_t B - N^i \partial_i B). \quad (4.3)$$

Let us now introduce $U(1)$ symmetry where the shift function transforms as

$$\delta_\alpha N_i(\mathbf{x}, t) = N(\mathbf{x}, t) \nabla_i \alpha(\mathbf{x}, t). \quad (4.4)$$

It is important to stress that as opposite to the case of pure Hořava-Lifshitz gravity the kinetic term is multiplied with B that is space-time dependent and hence it is not possible

to perform similar analysis as in [10]. This procedure frequently uses the integration by parts and the fact that covariant derivative annihilates metric tensor together with the crucial assumption that N depends on time only. Now due to the presence of B field we have to proceed step by step with the construction of the action invariant under (4.4). As the first step note that under (4.4) the kinetic term $S^{\text{kin}} = \frac{1}{\kappa^2} \int dt d^D \mathbf{x} \sqrt{g} K_{ij} \mathcal{G}^{ijkl} K_{kl}$ transforms as

$$\delta_\alpha S^{\text{kin}} = -\frac{2}{\kappa^2} \int dt d^D \mathbf{x} \sqrt{g} N B K_{ij} \mathcal{G}^{ijkl} \nabla_i \nabla_j \alpha.$$

In order to compensate this variation of the action we introduce new scalar field ν that under (4.4) transforms as

$$\delta_\alpha \nu(t, \mathbf{x}) = \alpha(t, \mathbf{x}) \quad (4.5)$$

and add to the action following term

$$S_\nu^{(1)} = \frac{2}{\kappa^2} \int dt d^D \mathbf{x} \sqrt{g} N B K_{ij} \mathcal{G}^{ijkl} \nabla_i \nabla_j \nu. \quad (4.6)$$

Note that under (4.5) this term transforms as

$$\delta_\alpha S_\nu^{(1)} = \frac{2}{\kappa^2} \int dt d^D \mathbf{x} \sqrt{g} N B K_{ij} \mathcal{G}^{ijkl} \nabla_k \nabla_l \alpha - \frac{2}{\kappa^2} \int dt d^D \mathbf{x} \sqrt{g} N B \nabla_i \nabla_j \alpha \mathcal{G}^{ijkl} \nabla_k \nabla_l \nu.$$

so that we add the second term into the action

$$S_\nu^{(2)} = \frac{1}{\kappa^2} \int dt d^D \mathbf{x} \sqrt{g} N B \nabla_i \nabla_j \nu \mathcal{G}^{ijkl} \nabla_k \nabla_l \nu. \quad (4.7)$$

As a result, we find that $S^{\text{kin}} + S_\nu^{(1)} + S_\nu^{(2)}$ is invariant under (4.4) and (4.5).

As the next step we analyze the variation of the B -kinetic part of the action $S^{B\text{kin}} = -\frac{2\mu}{\kappa^2} \int dt d^D \mathbf{x} \sqrt{g} N \nabla_n B K$ under the variation (4.4)

$$\delta_\alpha S^{B\text{kin}} = \frac{2\mu}{\kappa^2} \int dt d^D \mathbf{x} \sqrt{g} \alpha \nabla^i (\nabla_i B K) + \frac{2\mu}{\kappa^2} \int dt d^D \mathbf{x} \sqrt{g} N \alpha \nabla_i \nabla_j (g^{ij} \nabla_n B). \quad (4.8)$$

We see that in order to cancel this variation it is appropriate to add following expression into the action

$$\begin{aligned} S^{\nu-B} = & -\frac{2}{\kappa^2} \mu \int dt d^D \mathbf{x} \sqrt{g} N \nu \nabla^i (\nabla_i B K) - \frac{2}{\kappa^2} \mu \int dt d^D \mathbf{x} \sqrt{g} N \nu \nabla_i \nabla_j [g^{ij} \nabla_n B] \\ & + \frac{2}{\kappa^2} \mu \int dt d^D \mathbf{x} \sqrt{g} N \nabla^k \nu \nabla_k B \nabla_i \nabla^i \nu. \end{aligned}$$

Then it is easy to see that $S^{B\text{kin}} + S^{\nu-B}$ is invariant under (4.4) and (4.5). Collecting all these results we find following $F(\hat{R})$ HL action that is invariant under (4.4) and (4.5)

$$\begin{aligned} S_{F(\hat{R})} = & \frac{1}{\kappa^2} \int dt d^D \mathbf{x} \left(\sqrt{g} N B (K_{ij} \mathcal{G}^{ijkl} K_{kl} - \mathcal{V}(g) - A) \right. \\ & + \sqrt{g} N F(A) - 2\mu \sqrt{g} N \nabla_n B K + 2\mu \partial_i B \sqrt{g} g^{ij} \partial_j N \\ & - 2\mu \int d^D \mathbf{x} dt \sqrt{g} \nu \nabla^i (\nabla_i B K) - 2\mu \int d^D \mathbf{x} dt \sqrt{g} N \nu \nabla_i \nabla_j (g^{ij} \nabla_n B) \\ & + 2\mu \int dt d^D \mathbf{x} \sqrt{g} N \nabla^i \nu \nabla_i B \nabla^j \nabla_j \nu \\ & \left. + 2 \int d^D \mathbf{x} \sqrt{g} N B K_{ij} \mathcal{G}^{ijkl} \nabla_k \nabla_l \nu + \int d^D \mathbf{x} \sqrt{g} B \nabla_i \nabla_j \nu \mathcal{G}^{ijkl} \nabla_k \nabla_l \nu \right). \quad (4.9) \end{aligned}$$

Note that we can write this action in suggestive form

$$S_{F(\tilde{R})} = \frac{1}{\kappa^2} \int dt d^D \mathbf{x} \left(\sqrt{g} N B ((K_{ij} + \nabla_i \nabla_j \nu) \mathcal{G}^{ijkl} (K_{kl} + \nabla_k \nabla_l \nu) - \mathcal{V}(g) - A) \right. \\ \left. + \sqrt{g} N F(A) - 2\mu \sqrt{g} N (\nabla_n B + \nabla^i \nu \nabla_i B) g^{ij} (K_{ji} + \nabla_j \nabla_i \nu) + 2\mu \partial_i B \sqrt{g} g^{ij} \partial_j N \right) \quad (4.10)$$

or in even more suggestive form by introducing

$$\bar{N}_i = N_i - N \nabla_i \nu, \quad \bar{K}_{ij} = \frac{1}{2N} (\partial_t g_{ij} - \nabla_i \bar{N}_j - \nabla_j \bar{N}_i) \quad (4.11)$$

so that

$$S_{F(\tilde{R})} = \frac{1}{\kappa^2} \int dt d^D \mathbf{x} \left(\sqrt{g} N B (\bar{K}_{ij} \mathcal{G}^{ijkl} \bar{K}_{kl} - \mathcal{V}(g) - A) \right. \\ \left. + \sqrt{g} N F(A) - 2\mu \sqrt{g} N \hat{\nabla}_n B g^{ij} \bar{K}_{ji} + 2\mu \partial_i B \sqrt{g} g^{ij} \partial_j N \right) \quad (4.12)$$

is formally the same as the $F(\tilde{R})$ Hořava-Lifshitz gravity action. Clearly this action is invariant under arbitrary $\alpha = \alpha(t, \mathbf{x})$. Moreover, such an introduction of $U(1)$ symmetry is trivial and does not modify the physical properties of the theory. This is nicely seen from the fact that ν appears in the action in the combination with N_i through \bar{N}_i where ν plays the role of the Stückelberg field. In order to get physical content of given symmetry we follow [10] and [13] and introduce following term into action

$$S^{\nu,k} = \frac{1}{\kappa^2} \int dt d^D \mathbf{x} \sqrt{g} B \mathcal{G}(g_{ij}) (\mathcal{A} - a), \quad (4.13)$$

where

$$a = \dot{\nu} - N^i \nabla_i \nu + \frac{N}{2} \nabla^i \nabla_i \nu, \quad (4.14)$$

where $\dot{X} \equiv \frac{dX}{dt}$. Note a transforms under α variation as

$$a'(t, \mathbf{x}) = a(t, \mathbf{x}) + \dot{\alpha}(t, \mathbf{x}) - N^i(t, \mathbf{x}) \nabla_i \alpha(t, \mathbf{x}).$$

Then it is natural to suppose that \mathcal{A} transforms under α -variation as

$$\mathcal{A}'(t, \mathbf{x}) = \mathcal{A}(t, \mathbf{x}) + \dot{\alpha}(t, \mathbf{x}) - N^i(t, \mathbf{x}) \nabla_i \alpha(t, \mathbf{x}) \quad (4.15)$$

so that we immediately see that $\mathcal{A} - a$ is invariant under α -variation. The function \mathcal{G} can generally depend on arbitrary combinations of metric g and matter field and we only demand that it should be invariant under foliation preserving diffeomorphism (2.7) and under (4.4) and (4.5). For our purposes it is, however, sufficient to restrict ourselves to the models where \mathcal{G} depends on the spatial curvature R only. Now one observes that the equation of motion for \mathcal{A} implies the constraint

$$B \mathcal{G}(R) = 0 \quad (4.16)$$

that for non-zero B implies the condition $\mathcal{G}(R) = 0$. Note that this condition is crucial for elimination of the scalar graviton when we study fluctuations around flat background. We demonstrate this important result in the next section.

Finally note that it is possible to integrate out B and A fields from the actions (4.12) and (4.13) that leads to

$$S_{F(\bar{R})} = \frac{1}{\kappa^2} \int dt d^D \mathbf{x} \sqrt{g} N F(\bar{R}),$$

$$\bar{R} = \bar{K}_{ij} \mathcal{G}^{ijkl} \bar{K}_{kl} - \mathcal{V}(g) + \frac{2\mu}{\sqrt{g}N} \{ \partial_t (\sqrt{g} \bar{K}) - \partial_i (\sqrt{g} N^i \bar{K}) \} + \frac{1}{N} \mathcal{G}(\mathcal{A} - a). \quad (4.17)$$

This finishes the construction of $U(1)$ invariant $F(\bar{R})$ HL theory action.

4.1 Lagrangian for the Scalar Field

Now we extend above analysis to the action for the matter field with the following general form of the scalar field action

$$S_{\text{matt}} = - \int dt d^D \mathbf{x} \sqrt{g} N X, \quad (4.18)$$

where

$$X = -(\nabla_n \phi)^2 + F(g^{ij} \partial_i \phi \partial_j \phi). \quad (4.19)$$

where $F(x) = X + \sum_{n=2}^{\infty} X^n$ and where we defined

$$\nabla_n \phi = \frac{1}{N} (\partial_t \phi - N^i \partial_i \phi). \quad (4.20)$$

Note that this general form of the scalar field action is consistent with the anisotropy of target space-time as was shown in [24].

Now we try to extend above action in order to make it invariant under (4.4). Note that under this variation the scalar field action (4.18) transforms as

$$\delta_\alpha S_{\text{matt}} = -2 \int dt d^D \mathbf{x} \sqrt{g} N \nabla^i \alpha \nabla_i \phi \nabla_n \phi \quad (4.21)$$

using

$$\delta_\alpha X = 2 \nabla^i \alpha \nabla_i \phi \nabla_n \phi. \quad (4.22)$$

We compensate the variation (4.21) by introducing additional term into action

$$S_{\text{matt}-\nu} = -2 \int dt d^D \mathbf{x} \sqrt{g} \nu N \nabla^i (\nabla_i \phi \nabla_n \phi) + \int dt d^D \mathbf{x} \sqrt{g} \nabla^i \nu \nabla^j \nu \nabla_i \phi \nabla_j \phi \quad (4.23)$$

Then the action (4.23) transforms under (4.4) and (4.5) as

$$\delta_\alpha S_{\text{matt}-\nu} = 2 \int dt d^D \mathbf{x} \sqrt{g} N \nabla^i \alpha \nabla_i \phi \nabla_n \phi \quad (4.24)$$

that compensates the variation (4.21).

In the same way one can analyze more general form of the scalar action

$$S_{\text{matt}} = - \int dt d^D \mathbf{x} \sqrt{g} N K(X). \quad (4.25)$$

In order to find the generalization of the action (4.25) which is invariant under (4.4) we introduce two auxiliary fields C , D and write the action (4.25) as

$$S_{\text{matt}} = - \int dt d^D \mathbf{x} \sqrt{g} N [K(C) + D(X - C)]. \quad (4.26)$$

Clearly this action transforms under (4.4) as

$$\delta_\alpha S_{\text{matt}} = - \int dt d^D \mathbf{x} \sqrt{g} N D \delta_\alpha X = -2 \int dt d^D \mathbf{x} \sqrt{g} N D \nabla^i \alpha \nabla_i \phi \nabla_n \phi, \quad (4.27)$$

where relation (4.22) is used. It is easy to see that the variation of the following term

$$S_{\text{matt}-\nu} = 2 \int dt d^D \mathbf{x} \sqrt{g} N D \nabla^i \nu \nabla_i \phi \nabla_n \phi + \int dt d^D \mathbf{x} \sqrt{g} D \nabla^i \nu \nabla^j \nu \nabla_i \phi \nabla_j \phi \quad (4.28)$$

compensates the variation (4.27). Finally note that (4.26) together with (4.28) can be written in more elegant form

$$S_{\text{matt}} = - \int dt d^D \mathbf{x} \sqrt{g} N [K(C) + D(\bar{X} - C)] = - \int dt d^D \mathbf{x} \sqrt{g} N K(\bar{X}), \quad (4.29)$$

where

$$\begin{aligned} \bar{X} &= -(\bar{\nabla}_n \phi)^2 + F(g^{ij} \partial_i \phi \partial_j \phi), \\ \bar{\nabla}_n \phi &= \frac{1}{N}(\partial_t \phi - \bar{N}^i \nabla_i \phi) = \frac{1}{N}(\partial_t \phi - N^i \nabla_i + N \nabla^i \nu \nabla_i \phi). \end{aligned}$$

In this section we constructed $U(1)$ -invariant scalar field action in the form which closely follows the original construction presented in [10]. In the next section more elegant approach to the construction of $U(1)$ invariant $F(\bar{R})$ HL gravity and the scalar field action is given.

4.2 Alternative Definition of $U(1)$ Invariant $F(\bar{R})$ HL gravity

To begin with we note that under the (local) $U_\Sigma(1)$ symmetry, the shift function $N_i(\mathbf{x}, t)$ and $\nu(\mathbf{x}, t)$ are transformed as

$$N_i(\mathbf{x}, t) \rightarrow N_i(\mathbf{x}, t) + N(t) \nabla_i \alpha(\mathbf{x}, t), \quad \nu(\mathbf{x}, t) \rightarrow \nu(\mathbf{x}, t) + \alpha(\mathbf{x}, t). \quad (4.30)$$

Therefore the combination

$$\bar{N}_i(\mathbf{x}, t) \equiv N_i(\mathbf{x}, t) - N(t) \nabla_i \nu(\mathbf{x}, t), \quad (4.31)$$

is invariant under the transformation of the local $U_\Sigma(1)$ symmetry. Then if we replace N_i with \bar{N}_i , one can always obtain the model with the local $U_\Sigma(1)$ symmetry.

For example, the extrinsic curvature K_{ij} could be replaced by

$$K_{ij} \rightarrow \bar{K}_{ij} = \frac{1}{2N} (\partial_t g_{ij} - \nabla_i \bar{N}_j - \nabla_j \bar{N}_i) = K_{ij} + \frac{1}{2} (\nabla_i \nabla_j + \nabla_j \nabla_i) \nu. \quad (4.32)$$

Then it follows

$$\begin{aligned} S &= \frac{1}{2\kappa^2} \int dt d^D \mathbf{x} \sqrt{g} N B K_{ij} \mathcal{G}^{ijkl} K_{kl} \\ &\rightarrow \frac{1}{2\kappa^2} \int dt d^D \mathbf{x} \sqrt{g} N B \bar{K}_{ij} \mathcal{G}^{ijkl} \bar{K}_{kl} \\ &= \frac{1}{2\kappa^2} \int dt d^D \mathbf{x} \sqrt{g} \left\{ N B K_{ij} \mathcal{G}^{ijkl} K_{kl} + 2 N B K_{ij} \mathcal{G}^{ijkl} \nabla_k \nabla_l \nu + B \nabla_i \nabla_j \nu \mathcal{G}^{ijkl} \nabla_k \nabla_l \nu \right\} \end{aligned}$$

In the same way one can deal with following term

$$\begin{aligned} \nabla_\mu (n^\mu \nabla_\nu n^\mu - n^\nu \nabla_\nu n^\mu) &= \frac{1}{\sqrt{g} N} \left\{ \partial_t (\sqrt{g} K) - \partial_i (\sqrt{g} N^i K) \right\} \\ &\rightarrow \frac{1}{\sqrt{g} N} \left\{ \partial_t (\sqrt{g} \bar{K}) - \partial_i (\sqrt{g} N^i \bar{K}) \right\}. \end{aligned} \quad (4.34)$$

The above analysis tells that if we define a “curvature” by

$$\bar{R} = \bar{K}_{ij} \bar{K}^{ij} - \lambda \bar{K}^2 + \frac{2\mu}{\sqrt{g} N} \left\{ \partial_t (\sqrt{g} \bar{K}) - \partial_i (\sqrt{g} N^i \bar{K}) \right\} - \mathcal{V}(g_{ij}), \quad (4.35)$$

the action

$$S = \frac{1}{2\kappa^2} \int dt d^3 x \sqrt{g} N F(\bar{R}), \quad (4.36)$$

is invariant under the local $U_\Sigma(1)$ transformation (4.30). Finally, in order to obtain non-trivial symmetry we have to add to \bar{R} the expression $\mathcal{G}(g_{ij})(\mathcal{A}-a)$. Then the action derived here coincides with the action (4.17).

Note that the equation obtained by the variation of ν gives a constraint which kills the extra and problematic scalar mode appearing in the original Hořava gravity. For the action (4.36), the equation has the following form:

$$\begin{aligned} 0 &= \frac{N}{2} (\nabla_i \nabla_j + \nabla_j \nabla_i) (\bar{K}^{ij} F'(\bar{R})) - \lambda N \nabla^2 (\bar{K} F'(\bar{R})) \\ &\quad - 2\mu \nabla^2 (\partial_t F'(\bar{R}) - N^i \partial_i F'(\bar{R})) \\ &\quad + \frac{1}{\sqrt{g}} \left(\frac{d}{dt} (\sqrt{g} F'(\bar{R}) \mathcal{G}) - \nabla_i (N^i \sqrt{g} F'(\bar{R}) \mathcal{G}) - \frac{N}{2} \nabla^i \nabla_i (\sqrt{g} F'(\bar{R}) \mathcal{G}) \right) = 0. \end{aligned} \quad (4.37)$$

In the same way we can proceed with the action for scalar field ϕ (4.25). Instead of using step by step procedure performed in previous section one immediately makes the replacement $N^i \rightarrow \bar{N}^i = N^i - N \nabla^i \nu$ that leads to the action (4.29).

5. Hamiltonian Formalism of $U(1)$ Invariant $F(\tilde{R})$ HL Gravity

Let us again consider an action

$$\begin{aligned} S_{F(\tilde{R})} = & \frac{1}{\kappa^2} \int dt d^D \mathbf{x} \left(\sqrt{g} N B (\bar{K}_{ij} \mathcal{G}^{ijkl} \bar{K}_{kl} - \mathcal{V}(g) - A) \right. \\ & + \sqrt{g} N F(A) - 2\mu \sqrt{g} N \bar{\nabla}_n B \bar{K} + 2\mu \partial_i B \sqrt{g} g^{ij} \partial_j N \\ & \left. + \frac{1}{\kappa^2} \int dt d^D \mathbf{x} \sqrt{g} B \mathcal{G}(g_{ij}) (\mathcal{A} - a) \right). \end{aligned} \quad (5.1)$$

Now we perform the Hamiltonian analysis of given action. Note that the Hamiltonian analysis of $F(\tilde{R})$ HL gravity was performed previously in [16, 9, 18] and we generalize these works to the case of $U(1)$ invariant $F(\tilde{R})$ HL gravity.

From (5.1) we find the conjugate momenta

$$\begin{aligned} \kappa^2 \pi^{ij} &= \sqrt{g} B \mathcal{G}^{ijkl} \bar{K}_{kl} - \mu \sqrt{g} \hat{\nabla}_n B g^{ij}, \quad p_N \approx 0, \quad p^i \approx 0, \\ \kappa^2 p_B &= -2\mu \sqrt{g} \bar{K}, \quad p_A \approx 0, \quad p_{\mathcal{A}} \approx 0, \quad p_\nu = -\frac{1}{\kappa^2} \sqrt{g} N \mathcal{G}. \end{aligned}$$

From these relations the primary constraints are found

$$\Phi_1 : p_A(\mathbf{x}) \approx 0, \quad \Phi_2 : p_\nu(\mathbf{x}) + \frac{1}{\kappa^2} \sqrt{g} N \mathcal{G}(\mathbf{x}), \quad p_A(\mathbf{x}) \approx 0, \quad p^i(\mathbf{x}) \approx 0, \quad p_N \approx 0. \quad (5.2)$$

The Hamiltonian density is obtained in the form

$$\mathcal{H} = N \mathcal{H}_T + N^i \mathcal{H}_i,$$

where

$$\begin{aligned} \mathcal{H}_T = & \frac{\kappa^2}{\sqrt{g} B} \pi^{ij} g_{ik} g_{il} \pi^{kl} - \frac{\kappa^2}{\sqrt{g} B D} \pi^2 - \frac{\kappa^2 \pi p_B}{\sqrt{g} \mu D} \\ & + \frac{B \kappa^2}{4 \mu^2 \sqrt{g} D} (\lambda D - 1) p_B^2 - \frac{1}{\kappa^2} \sqrt{g} B (\mathcal{V}(g) - A) - \frac{1}{\kappa^2} \sqrt{g} F(A) + \frac{2\mu}{\kappa^2} \partial_i [\sqrt{g} g^{ij} \partial_j B] \\ & - 2\nu \nabla_i \nabla_j \pi^{ij} + \nu \nabla^i \nabla_i B + \frac{1}{2\kappa^2} \mathcal{G}(R) B \nabla^i \nabla_i \nu, \\ \mathcal{H}_i = & -2g_{il} \nabla_k \pi^{kl} + p_B \nabla_i B - \frac{1}{\kappa^2} \sqrt{g} B \mathcal{G}(R) \nabla_i \nu, \end{aligned} \quad (5.3)$$

where

$$\mathcal{G}_{ijkl} = \frac{1}{2} (g_{ik} g_{jl} + g_{il} g_{jk}) - \frac{\lambda}{D\lambda - 1} g_{ij} g_{kl}. \quad (5.4)$$

According to the general analysis of constraint systems⁵ we should consider following Hamiltonian

$$H = \int d^D \mathbf{x} (N \mathcal{H}_T + N^i \mathcal{H}_i + v^1 \Phi_1 + v^2 \Phi_2 + v^A p_A + w_i p^i + w_N p_N) - \frac{1}{\kappa^2} \int d^D \mathbf{x} \sqrt{g} \mathcal{G}(R) \mathcal{A},$$

⁵For review, see [25].

where $v^{1,2}$, v^A , w_i , and w_N are Lagrange multipliers related to corresponding primary constraints. Now the requirement of the conservation of the primary constraints $p_N \approx 0$, $p_i(\mathbf{x}) \approx 0$, $\Phi_{1,2}(\mathbf{x}) \approx 0$ and $p_A(\mathbf{x}) \approx 0$ implies following secondary ones

$$\begin{aligned}\partial_t \Phi_1 &= \{\Phi_1, H\} = -\frac{1}{\kappa^2} B \sqrt{g} \mathcal{G} \equiv -\Phi_1^{II} \approx 0, \\ \partial_t p_N &= \{p_N, H\} = -\int d^D \mathbf{x} \mathcal{H}_T \equiv -\mathbf{T} \approx 0, \\ \partial_t p_i &= \{p_i, H\} = -\mathcal{H}_i \approx 0, \\ \partial_t p_A(\mathbf{x}) &= -\sqrt{g} B + \sqrt{g} F'(A) \equiv \sqrt{g} G_A \approx 0.\end{aligned}\tag{5.5}$$

Finally we determine the time evolution of Φ_2 . Using

$$\{R(\mathbf{x}), \pi^{ij}(\mathbf{y})\} = -R^{ij}(\mathbf{x})\delta(\mathbf{x} - \mathbf{y}) + \nabla^i \nabla^j \delta(\mathbf{x} - \mathbf{y}) - g^{ij} \nabla_k \nabla^k \delta(\mathbf{x} - \mathbf{y})$$

we find

$$\partial_t \Phi_2 = \{\Phi_2, H\} \approx N \frac{d\mathcal{G}}{dR} \Phi_2^{II},\tag{5.6}$$

where the secondary constraint Φ_2^{II} takes the form

$$\Phi_2^{II} = -2R_{ij}\pi^{ij} + \frac{2}{D}R\pi - \frac{2}{D}B\nabla_k \nabla^k \left(\frac{\pi}{B}\right) + \frac{B}{\mu D}(Rp_B - (1-D)\nabla_i \nabla^i p_B).\tag{5.7}$$

Note that in the calculation of (5.6) we used the fact that

$$\{p_\nu, H\} = -N\nabla^i \mathcal{H}_i + \frac{1}{\kappa^2} \nabla_i (\sqrt{g} N^i \mathcal{G}) + \frac{N}{2\kappa^2} \nabla^i \nabla_i (\sqrt{g} \mathcal{G}) \sim \Phi_1^{II} \approx 0\tag{5.8}$$

using that the right hand of this equation is proportional to Φ_1^{II} or its covariant derivatives and hence it vanishes on constraint surface $\Phi_1^{II} \approx 0$. In fact, we previously simplified the equation (5.7) where all terms proportional to the constraint Φ_1^{II} were ignored.

Now one can summarize the results. We have the primary constraints $p_N \approx 0$, $p_i \approx 0$, $p_A \approx 0$, $p_\nu \approx 0$, $\Phi_1^{II} \approx 0$, $\Phi_2^{II} \approx 0$, $\mathcal{H}_i \approx 0$, $\mathbf{T} \approx 0$. Then the total Hamiltonian takes the form

$$\begin{aligned}H &= N\mathbf{T} + v^N p_N \\ &+ \int d^D \mathbf{x} (N^i \mathcal{H}_i + v^i p_i + v^A p_A + v^\nu p_\nu + v_{II}^1 \Phi_1^{II} + v_{II}^2 \Phi_2^{II} + v^A p_A + v^G G_A),\end{aligned}$$

where we included an expression $\Phi_1^{II} \mathcal{A}$ into the expression $v_{II}^1 \Phi_1^{II}$.

Let us analyze the consistency of the secondary constraints with the time development of the system for the case $F''(A) \neq 0$. To begin with note that one can write \mathcal{H}_i as $\mathcal{H}_i = -2g_{ik} \nabla_l \pi^{kl} + p_B \partial_i B + p_\nu \partial_i \nu - \Phi_1^{II} \partial_i \nu$ so that it is natural to consider as the smeared form of diffeomorphism constraint the following expression

$$\mathbf{T}_S(M^i) = \int d^D \mathbf{x} M^i (-2g_{ik} \nabla_l \pi^{kl} + p_B \partial_i B + p_\nu \partial_i \nu)\tag{5.9}$$

that is the generator of spatial diffeomorphism. Then using the fact that the total Hamiltonian is manifestly invariant under spatial diffeomorphism one finds that \mathbf{T}_S is preserved

on the constraint surface. In case of the constraint p_ν , using (5.8) and also the fact that $\{p_\nu, \Phi_1^{II}\} = \{p_\nu, \Phi_2^{II}\} = 0$ it follows that this constraint is preserved during the time evolution of the system and also that it is the first class constraint.

Let us now consider the constraint Φ_1^{II} . Its time development is governed by equation

$$\partial_t \Phi_1^{II} = \{\Phi_1^{II}, H_T\} \approx N \frac{d\mathcal{G}}{dR} \Phi_2^{II} + \int d^D \mathbf{x} v_{II}^2(\mathbf{x}) \{\Phi_1^{II}, \Phi_2^{II}(\mathbf{x})\}. \quad (5.10)$$

To proceed further we note that from the structure of the constraint Φ_1^{II} and Φ_2^{II} we clearly have

$$\{\Phi_1^{II}(\mathbf{x}), \Phi_2^{II}(\mathbf{y})\} \neq 0. \quad (5.11)$$

Say differently, Φ_1^{II} together with Φ_2^{II} are the second class constraints. Then the requirement that the right side of the equation (5.10) has to vanish implies that the Lagrange multiplier v_{II}^2 has to be equal to zero. On the other hand the time evolution of Φ_2^{II} is governed by equation

$$\partial_t \Phi_2^{II} = N \{\Phi_2^{II}, \mathbf{T}\} + \int d^D \mathbf{x} (v^G(\mathbf{x}) \{\Phi_2, G(\mathbf{x})\} + v_{II}^1(\mathbf{x}) \{\Phi_2^{II}, \Phi_1^{II}(\mathbf{x})\}). \quad (5.12)$$

Note also that $\{\Phi_2^{II}(\mathbf{x}), G_A(\mathbf{y})\} \neq 0$ as follows from (5.5) and from (5.7). On the other hand the time evolution of G_A is governed by equation

$$\partial_t G_A = \{G_A, H_T\} = N \{G_A, \mathbf{T}\} + \int d^D \mathbf{x} (v_{II}^2(\mathbf{x}) \{G_A, \Phi_2^{II}(\mathbf{x})\} + v^A(\mathbf{x}) \{G_A, p_A(\mathbf{x})\}) \quad (5.13)$$

Since $\{p_A(\mathbf{x}), G_A(\mathbf{y})\} = -\sqrt{g}F''(A)\delta(\mathbf{x} - \mathbf{y})$ and since $v_{II}^2 = 0$ we see that requirement of the vanishing of the right side of (5.13) determines v^A as a function of canonical variables. On the other hand the requirement of the preservation of the constraint p_A

$$\partial_t p_A = \{p_A, H_T\} \approx \int d^D \mathbf{x} v^G \{p_A, G_A(\mathbf{x})\} = 0 \quad (5.14)$$

implies $v^G = 0$. However using this equation in (5.12) we can again determine the Lagrange multiplier v_{II}^1 as function of canonical variables. At this place we see that the requirement of the preservation of the constraints $\Phi_{1,2}^{II}$, p_A , G_A determines the corresponding Lagrange multipliers $v_{II}^{1,2}$, v^A , v^G and consequently no new constraints have to be imposed.

Now we are ready to determine the number of physical degrees of freedom. To do this note that there are $D(D+1)$ gravity phase space variables g_{ij} , π^{ij} , $2D$ variables N_i, p^i , 2 variables \mathcal{A}, p_A , 2 variables B, p_B , 2 variables A, p_A and 2 variables ν, p_ν . In summary the total number of degrees of freedom is $N_{\text{D.o.f}} = D^2 + 3D + 8$. On the other hand we have D first class constraints $\mathcal{H}_i \approx 0$, D first class constraints $p_i \approx 0$, 2 first class constraints $p_\nu \approx 0$, $p_A \approx 0$ and four second class constraints Φ_1^{II} , Φ_1^{III} , p_A , G_A . Then there are $N_{\text{f.c.c.}} = 2D + 2$ first class constraints and $N_{\text{s.c.c.}} = 4$ second class constraints. The number of physical degrees of freedom is [25]

$$N_{\text{D.o.f.}} - 2N_{\text{f.c.c.}} - N_{\text{s.c.c.}} = (D^2 - D - 2) + 1 \quad (5.15)$$

that exactly corresponds to the number of the phase space physical degrees of freedom of $D + 1$ dimensional $F(R)$ gravity. For example, for $D = 3$ the equation (5.15) gives 4 phase space degrees of freedom corresponding massless graviton and 2 phase degrees of freedom corresponding to the scalar. Note also that there is still global Hamiltonian constraint that has to be imposed and also that all second class constraints have to be solved. Solving of $p_A = 0$, $G_A = 0$ one can express A as a function of B , at least in principle. Unfortunately solving the second class constraints Φ_1^{II} , Φ_2^{II} is very difficult in the full generality. On the other hand, it is easy to see that in linearized approximation these constraints can be solved as $h = 0$, $\pi = 0$ where h is the trace part of the metric fluctuation and π is its conjugate momenta.

The previous analysis was valid for the case when $F''(A) \neq 0$. Let us now discuss the second case when $F'(A) = 1$ ⁶. For $F'(A) = 1$ we see that the Poisson bracket between p_A and G_A is zero. Then the equation (5.13) implies additional constraint (Note that $v_{II}^2 = 0$)

$$\partial_t G_A = \{G_A, H_T\} = N \{G_A, \mathbf{T}\} \equiv \frac{N\kappa^2}{\sqrt{g}\mu D} G_A^{II}, \quad (5.16)$$

where

$$G_A^{II} = \pi - (\lambda D - 1) \frac{B p_B}{2\mu}. \quad (5.17)$$

Due to the fact that $\{p_A, G_A\} = 0$ one finds that $p_A \approx 0$ is the first class constraint. On the other hand we have following non-zero Poisson bracket

$$\{G_A(\mathbf{x}), G_A^{II}(\mathbf{y})\} = \frac{(\lambda D - 1)}{2\mu} B(\mathbf{x}) \delta(\mathbf{x} - \mathbf{y}) \quad (5.18)$$

that implies that G_A and G_A^{II} are the second class constraints. The analysis of the remaining constraints is the same as in case $F''(A) \neq 0$ so that we have following set of constraints. There are D first class constraints $\mathcal{H}_i \approx 0$, D first class constraints $p_i \approx 0$, 3 first class constraints $p_\nu \approx p_A \approx 0$, $p_A \approx 0$ and four second class constraints Φ_1^{II} , Φ_1^{III} , G_A , G_A^{II} . Then we have $N_{f.c.c.} = 2D + 3$ first class constraints and $N_{s.c.c.} = 4$ second class constraints. The first class constraint $p_A = 0$ can be eliminated with the gauge fixing condition $A = 1$. Solving the constraint $G_A = 0$ we obtain $B = 1$ while solving the constraint $G_A^{II} = 0$ we find

$$p_B = \frac{2\mu\pi}{\lambda D - 1}. \quad (5.19)$$

Inserting this result into \mathcal{H}_T given in (5.3) one gets that it takes the form

$$\mathcal{H}_T = \frac{\kappa^2}{\sqrt{g}} \left(\pi^{ij} g_{ik} g_{jl} \pi^{kl} - \frac{\lambda}{\lambda D - 1} \right) \pi^2 - \frac{1}{\kappa^2} \sqrt{g} \mathcal{V}(g) - 2\nu \nabla_i \nabla_j \pi^{ij} + \frac{1}{2\kappa^2} \sqrt{g} \mathcal{G}(R) \nabla^i \nabla_i \nu,$$

that corresponds to the Hamiltonian constraint of non-relativistic covariant HL gravity whose explicit form can be found in [14]. In the same way we insert (5.19) into (5.7) and we find

$$\Phi_2^{II} = -2R_{ij} \pi^{ij} + 2 \frac{\lambda}{D\lambda - 1} R\pi + \frac{1 - \lambda}{\lambda D - 1} \nabla_i \nabla^i \pi$$

⁶Generally we could have $F'(A) = C$ for constant C . However we show below that in this case $B = C$ and consequently C can be eliminated by redefinition of κ .

which again coincides with the constraint found in [14]. In other words in case when $F''(A) = 0$ the Hamiltonian structure of $U(1)$ invariant $F(\bar{R})$ gravity coincides with the Hamiltonian structure of non-relativistic covariant HL gravity. This result can be considered as a nice confirmation of our procedure.

6. Study of Fluctuations around Flat Background in $U(1)$ Invariant $F(\tilde{R})$ HL Gravity

Let us analyze the spectrum of fluctuations in case of $U(1)$ invariant $F(\tilde{R})$ HL gravity for the special case $\mu = 0$. For simplicity we assume that $F(\tilde{R})$ Hořava-Lifshitz gravity has flat space-time as its solution with the background values of the fields

$$g_{ij}^{(0)} = \delta_{ij}, \quad N^{(0)} = 1, \quad N_i^{(0)} = 0, \quad \mathcal{A}^{(0)} = 0, \quad \nu^{(0)} = 0. \quad (6.1)$$

Note that for the flat background the equation of motion for B and A takes the form

$$\mathcal{V}(g^{(0)}) - A^{(0)} = 0, \quad B^{(0)} - F'(A^{(0)}) = 0 \quad (6.2)$$

that implies that $A^{(0)}, B^{(0)}$ are constants. In order to find the spectrum of fluctuations we expand all fields up to linear order around this background

$$\begin{aligned} g_{ij} &= \delta_{ij} + \kappa h_{ij}, \quad N_i = \kappa n_i, \quad N = 1 + \kappa n, \\ A &= A^{(0)} + \kappa a, \quad B = B^{(0)} + \kappa b, \quad \mathcal{A} = \mathcal{A}^{(0)} + \kappa \tilde{\mathcal{A}}, \quad \nu = \nu^{(0)} + \kappa \tilde{\nu}. \end{aligned}$$

Since n depends on t only, its equation of motion gives one integral constraint. This constraint does not affect the number of local degrees of freedom. For that reason it is natural to consider the equation of motion for h_{ij} , n_i and ν only. We further decompose the field h_{ij} and n_i into their irreducible components

$$h_{ij} = s_{ij} + \partial_i w_j + \partial_j w_i + \left(\partial_i \partial_j - \frac{1}{D} \delta_{ij} \partial^2 \right) M + \frac{1}{D} \delta_{ij} h, \quad (6.3)$$

where the scalar $h = h_{ii}$ is the trace part of h_{ij} while s_{ij} is symmetric, traceless and transverse

$$\partial^i s_{ij} = 0, \quad \partial^i = \delta^{ij} \partial_j \quad (6.4)$$

and w_i is transverse

$$\partial^i w_i = 0. \quad (6.5)$$

In the same way we decompose n_i

$$n_i = u_i + \partial_i C \quad (6.6)$$

with u_i transverse $\partial^i u_i = 0$. In what follows we fix the spatial diffeomorphism symmetry by fixing the gauge

$$w_i = 0, \quad M = 0. \quad (6.7)$$

We begin with the equation of motion for N_i

$$2\nabla_j(B\mathcal{G}^{ijkl}\bar{K}_{kl}) + B\mathcal{G}(R)\nabla_i\nu = 0. \quad (6.8)$$

and for ν

$$2\nabla_i\nabla_j(BN\mathcal{G}^{ijkl}\bar{K}_{kl}) + \frac{1}{\sqrt{g}}\left(\frac{d}{dt}(\sqrt{g}B\mathcal{G}) - \nabla_i(\sqrt{g}BN^i\mathcal{G}) - \frac{1}{2}\nabla_i\nabla^i(\sqrt{g}BN\mathcal{G})\right) = 0.$$

Combining these equations and using the equation of motion for \mathcal{A} one obtains

$$\mathcal{G}(R) = 0 \quad (6.9)$$

Then

$$B\frac{d}{dt}\mathcal{G} - BN^i\nabla_i\mathcal{G} - \frac{N}{2}(B\nabla_i\nabla^i\mathcal{G} + 2\nabla_iB\nabla^i\mathcal{G}) = 0. \quad (6.10)$$

Further, in the linearized approximation the equation of motion (6.8) takes the form

$$-2B_0(1-\lambda)\partial^i(\partial_k\partial^k C) + B_0\partial_k\partial^k u^i + \frac{1}{D}(1-D\lambda)\partial^i\dot{h} + 2(1-\lambda)B_0\partial^i(\partial_j\partial^j\tilde{\nu}) = 0 \quad (6.11)$$

using the fact that $\mathcal{G}(R_0) = 0$ and also $\partial^i s_{ij} = \partial^j s_{ij} = 0$ and $\delta^{ij}s_{ji} = 0$. Let us now focus on the solution of the constraint $\mathcal{G}(R) = 0$ in the linearized approximation. Let $R_0^{(D)}$ is the solution of the equation of motion and let us consider the perturbation around this equation. These perturbations have to obey the equation

$$\mathcal{G}(R_0 + \delta R) = \mathcal{G}(R_0) + \frac{d\mathcal{G}}{dR}\delta R = \frac{d\mathcal{G}}{dR}(R_0)\delta R = 0.$$

We see that in order to eliminate the scalar graviton we have to demand that $\frac{d\mathcal{G}}{dR}(R_0) \neq 0$. To proceed further note that

$$\delta R_{ij} = \frac{1}{2}[\nabla_i^{(0)}\nabla^{(0)k}h_{jk} + \nabla_j^{(0)}\nabla^{k(0)}h_{ik} - \nabla_k^{(0)}\nabla^{k(0)}h_{ij} - \nabla_i^{(0)}\nabla_j^{(0)}h] \quad (6.12)$$

where $\nabla^{(0)}$ is the covariant derivative calculated using the background metric $g_{ij}^{(0)}$. Then in the flat background it follows

$$\delta R = \frac{\kappa}{D}(1-D)\partial^k\partial_k h$$

so that the condition $\delta R = 0$ implies $\partial_k\partial^k h = 0$. Then $h = h(t)$. However, in this case the fluctuation mode does not obey the boundary conditions that we implicitly assumed. Explicitly we demand that all fluctuations vanish at spatial infinity. For that reason one should demand that $h = 0$. Then it is easy to see that the equation (6.10) is trivially solved. In the linearized approximation the equation of motion for g_{ij} takes the form

$$\begin{aligned} &\frac{1}{2}(\ddot{s}_{ij} + \frac{1}{D}\delta_{ij}(1-\lambda D)\ddot{h} - \partial_i\dot{u}_j - 2\partial_i\partial_j\dot{C} + 2\partial^i\partial^j\tilde{\nu} - 2\lambda\delta^{ij}\partial_k\partial^k\tilde{\nu} + 2\lambda\delta_{ij}\partial_k\partial^k\dot{C}) \\ &- \frac{\delta\mathcal{V}_2}{g_{ij}} + \frac{d\mathcal{G}}{dR}[\partial^i\partial^j(\tilde{\mathcal{A}} - \tilde{a}) + \delta^{ij}\partial_k\partial^k(\tilde{\mathcal{A}} - \tilde{a})] = 0 \end{aligned} \quad (6.13)$$

where

$$\tilde{a} = \frac{d\tilde{\nu}}{dt} + \frac{1}{2}\partial_k\partial^k\tilde{\nu}. \quad (6.14)$$

Note that we have not fixed the $U(1)$ gauge symmetry yet. It turns out that it is natural to fix it as

$$\nu = 0. \quad (6.15)$$

Then the trace of the equation (6.13) is equal to

$$(1 - \lambda D)\partial_k\partial^k\dot{C} + \delta^{ij}\frac{\delta\mathcal{V}_2}{\delta g_{ij}} - \frac{d\mathcal{G}}{dR}(1 - D)\partial_i\partial^i\tilde{\mathcal{A}} = 0. \quad (6.16)$$

Let us again consider the equation of motion (6.11) and take its ∂^i . Then using the fact that $\partial^i u_i = 0$ one gets the condition $C = f(t)$ that again with suitable boundary conditions implies $C = 0$. However then inserting this result in (6.11) we find

$$\partial_k\partial^k u_i = 0 \quad (6.17)$$

that also implies $u_i = 0$.

To proceed further we now assume that $\mathcal{V}_2 = -R$ so that $\frac{\delta\mathcal{V}_2}{\delta g_{ij}} = -\frac{1}{2}\partial_k\partial^k s_{ij} - \frac{D-2}{2D}(\partial_i\partial_j - \delta_{ij}\partial^k\partial_k)h$. Clearly the trace of this equation is proportional to h and hence it implies following equation for \tilde{a}

$$\partial_k\partial^k\tilde{\mathcal{A}} = 0. \quad (6.18)$$

Imposing again the requirement that $\tilde{\mathcal{A}}$ vanishes at spatial infinity we find that the only solution of given equation is $\tilde{\mathcal{A}} = 0$. Finally the equation of motion for g_{ij} gives following result

$$\ddot{s}_{ij} + \partial_k\partial^k s_{ij} = 0. \quad (6.19)$$

In other words, it is demonstrated that under assumption that $U(1)$ invariant $F(\bar{R})$ HL gravity has flat space-time as its solution it follows that the perturbative spectrum contains the transverse polarization of the graviton only. Clearly, this result may be generalized for general version of theory with arbitrary parameter μ .

Finally we consider the linearized equations of motion for A and B . In case of A one gets

$$-b + F''(A_0)a = 0 \quad (6.20)$$

while in case of B we obtain

$$-\frac{d\mathcal{V}}{dR}(R_0)\delta R - a = 0 \quad (6.21)$$

Using the fact that $\delta R \sim h = 0$ we get from (6.21) and from (6.20)

$$a = b = 0. \quad (6.22)$$

In other words there are no fluctuations corresponding to the scalar fields A and B . One can compare this situation with the conventional $F(R)$ gravity where the mathematical equivalence of the theory with the Brans-Dicke theory implies the existence of propagating scalar degrees of freedom. In our case, however, the fact that $U(1)$ invariant $F(\bar{R})$ HL

gravity is invariant under the foliation preserving diffeomorphism allows us to consider theory without kinetic term for B ($\mu = 0$).

Thus, it seems $U(1)$ extension of $F(R)$ HL gravity may lead to solution of the problem of scalar graviton.

7. Cosmological Solutions of $U(1)$ Invariant $F(\bar{R})$ HL gravity

Let us investigate the FRW cosmological solutions for the theory described by action (4.36). Spatially-flat FRW metric is now assumed

$$ds^2 = -N^2 dt^2 + a^2(t) \sum_{i=1}^3 (dx^i)^2, \quad (7.1)$$

and we choose the gauge fixing condition for the local $U_\Sigma(1)$ symmetry as follows,

$$\nu = 0. \quad (7.2)$$

In the FRW metric (7.1), one gets $N_i = 0$ and \bar{R} and \bar{K}_{ij} only depend on the cosmological time t . Therefore, the constraint equation (4.37) can be satisfied trivially. For the metric (7.1) and the gauge fixing condition, the scalar \bar{R} is given by

$$\bar{R} = \frac{3(1 - 3\lambda + 6\mu)H^2}{N^2} + \frac{6\mu}{N} \frac{d}{dt} \left(\frac{H}{N} \right). \quad (7.3)$$

The second FRW equation can be obtained by varying the action(4.36) with respect to the spatial metric g_{ij} , which yields

$$\begin{aligned} 0 = & F(\bar{R}) - 2(1 - 3\lambda + 3\mu) \left(\dot{H} + 3H^2 \right) F'(\bar{R}) - 2(1 - 3\lambda) \dot{\bar{R}} F''(\bar{R}) + \\ & + 2\mu \left(\dot{\bar{R}}^2 F^{(3)}(\bar{R}) + \ddot{\bar{R}} F''(\bar{R}) \right) + \kappa^2 p_m. \end{aligned} \quad (7.4)$$

Here the contribution from the matter is included. p_m expresses the pressure of a perfect fluid that fills the Universe, and $N = 1$. Note that this equation becomes the usual second FRW equation for conventional $F(R)$ gravity by setting the constants $\lambda = \mu = 1$. The variation with respect to N yields the following global constraint

$$0 = \int d^3x \left[F(\bar{R}) - 6(1 - 3\lambda + 3\mu)H^2 - 6\mu\dot{H} + 6\mu H \dot{\bar{R}} F''(\bar{R}) - \kappa^2 \rho_m \right]. \quad (7.5)$$

By assuming the ordinary conservation equation for the matter fluid $\dot{\rho}_m + 3H(\rho_m + p_m) = 0$, and integrating Eq. (7.4),

$$0 = F(\bar{R}) - 6 \left[(1 - 3\lambda + 3\mu)H^2 + \mu\dot{H} \right] F'(\bar{R}) + 6\mu H \dot{\bar{R}} F''(\bar{R}) - \kappa^2 \rho_m - \frac{C}{a^3}, \quad (7.6)$$

where C is an integration constant, taken to be zero, according to the constraint equation (7.5). Eq. (7.6) corresponds to the first FRW equation. Hence, starting from a given $F(\bar{R})$ function, and solving Eqs. (7.4) and (7.5), FRW cosmological solution can be obtained.

Note that the obtained equations (7.4) and (7.5) are identical with the corresponding equations in the $F(R)$ -gravity (for recent review of observational aspects of such theory see [20]) based on the original Hořava gravity. Hence, $U(1)$ extension does not influence the FRW cosmological dynamics.

Let us consider the theory which admits a de Sitter universe solution. We now neglect the matter contribution by putting $p_m = \rho_m = 0$. Then by assuming $H = H_0$, Eq. (7.6) gives

$$0 = F(3(1 - 3\lambda + 6\mu)H_0^2) - 6(1 - 3\lambda + 3\mu)H_0^2 F'(3(1 - 3\lambda + 6\mu)H_0^2), \quad (7.7)$$

as long as the integration constant vanishes ($C = 0$). We now consider the following model:

$$F(\bar{R}) \propto \bar{R} + \beta \bar{R}^2 + \gamma \bar{R}^3. \quad (7.8)$$

Then Eq. (7.7) becomes

$$0 = H_0^2 \left\{ 1 - 3\lambda + 9\beta(1 - 3\lambda + 6\mu)(1 - 3\lambda + 2\mu)H_0^2 + 9\gamma(1 - 3\lambda + 6\mu)^2(5 - 15\lambda + 12\mu)H_0^4 \right\}, \quad (7.9)$$

which has the following two non-trivial solutions,

$$H_0^2 = -\frac{(1 - 3\lambda + 2\mu)\beta}{2(1 - 3\lambda + 6\mu)(5 - 15\lambda + 12\mu)\gamma} \left(1 \pm \sqrt{1 - \frac{4(1 - 3\lambda)(5 - 15\lambda + 12\mu)\gamma}{9(1 - 3\lambda + 2\mu)^2\beta^2}} \right), \quad (7.10)$$

as long as the r.h.s. is real and positive. If

$$\left| \frac{4(1 - 3\lambda)(5 - 15\lambda + 12\mu)\gamma}{9(1 - 3\lambda + 2\mu)^2\beta^2} \right| \ll 1, \quad (7.11)$$

one of the two solutions is much smaller than the other solution. Then one may regard that the larger solution corresponds to the inflation in the early universe and the smaller one to the late-time acceleration.

More examples of $F(\bar{R})$ theory which can contain more than one dS solution, such that inflation and dark energy epochs can be explained under the same mechanism Ref. [26] may be considered. First of all, as generalization of the model (7.8), a general polynomial function may be discussed

$$F(\bar{R}) = \sum_{n=1}^m \alpha_n \bar{R}^n, \quad (7.12)$$

Here α_n are coupling constants. Using the equation (7.7), it yields the algebraic equation,

$$0 = \sum_{n=1}^m \alpha_n \bar{R}_0^n - 2 \frac{1 - 3\lambda + 3\mu}{1 - 3\lambda + 6\mu} \bar{R}_0 \sum_{n=1}^m n \alpha_n \bar{R}_0^{n-1}. \quad (7.13)$$

By a qualitative analysis, one can see that the number of positive real roots, i.e., of the de Sitter points, depends completely on the sign of the coupling constants α_n . Then, by a proper choice, $F(\bar{R})$ gravity can well explain dark energy and inflationary epochs in

a unified natural way. Even it could predict the existence of more than two accelerated epochs, which could resolve the coincidence problem.

Let us now consider an explicit example

$$F(\bar{R}) = \frac{\bar{R}}{\bar{R}(\alpha\bar{R}^{n-1} + \beta) + \gamma}, \quad (7.14)$$

where α, β, γ, n are constants. By introducing this function in (7.7), it is straightforward to show that for the function (7.14), there are several de Sitter solutions. In order to simplify this example, let us consider the case $n = 2$, where the equation (7.7) yields,

$$\gamma - 3\gamma\lambda - 3\beta H_0^2(1 - 3\lambda + 6\mu)^2 + 27\alpha H_0^4(-1 + 3\lambda - 4\mu)(1 - 3\lambda + 6\mu)^2 = 0. \quad (7.15)$$

The solutions are given by

$$H_0^2 = \frac{1}{18\alpha(1 - 3\lambda + 6\mu)^2(-1 + 3\lambda - 4\mu)} \left\{ \beta(1 - 3\lambda + 6\mu)^2 \pm \sqrt{(1 - 3\lambda + 6\mu)^2 [12\alpha\gamma(-1 + 3\lambda)(-1 + 3\lambda - 4\mu) + \beta^2(1 - 3\lambda + 6\mu)^2]} \right\}. \quad (7.16)$$

Then, by a proper choice of the free parameters of the model, two positive roots of the equation (7.15) are solutions. Hence, such a model can explain inflationary and dark energy epochs in unified manner.

8. Discussion

In summary, in this work we aimed to resolve (at least, partially) the inconsistency problems of the projectable HL gravity. First of all, it is demonstrated that some versions of $F(R)$ HL gravity may have stable de Sitter solution and unstable flat space solution. As a result, the spectrum analysis showing the presence of scalar graviton is not applied. The whole spectrum analysis should be redone for de Sitter background.

Second, $U(1)$ extension of $F(R)$ HL gravity is formulated in two alternative approaches. Hamiltonian structure of $U(1)$ invariant $F(\bar{R})$ gravity is investigated in all detail. The whole constraints system is derived and different particular cases corresponding to conditions for derivatives of function F are studied. It is demonstrated that in some cases the Hamiltonian structure of the theory coincides with the one of $U(1)$ invariant HL gravity that conforms consistency of our approach. The analysis of fluctuations of $U(1)$ invariant $F(\bar{R})$ HL gravity is performed. It is shown that like in case of $U(1)$ HL gravity the scalar graviton ghost does not emerge. This opens good perspectives for consistency of such class of models. It is also interesting that spatially-flat FRW equations for $U(1)$ invariant $F(\bar{R})$ gravity turn out to be just the same as for the one without $U(1)$ symmetry. This indicates that all (spatially-flat FRW) cosmological predictions of viable conventional $F(R)$ gravity are just the same as for its HL counterpart (with special parameters choice).

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